# Optimization and Discrete Mathematics

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***Integrity ★ Service ★ Excellence***



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# 2013 AFOSR Spring Review
# Optimization and Discrete Math

**PM: Fariba Fahroo**

**BRIEF DESCRIPTION OF PORTFOLIO:**

**Development of optimization and discrete mathematics for solving large, complex problems in support of future Air Force science and engineering, command and control systems, logistics, and battlefield management**

**LIST SUB-AREAS IN PORTFOLIO:**

- **Analysis based optimization**
- **Continuous and Discrete Search methods**
- **Dynamic, Stochastic and Simulation Optimization**
- **Combinatorial Optimization**

# Program Overview

**Optimization is at the heart of so many scientific fields and applications such as engineering, finance, mathematical biology, resource allocation, network theory, numerical analysis, control theory, decision theory, and so on.**

**A General Optimization Problem Formulation**

***Maximize (or Minimize) f(x)***

***Subject to***

$$g(x) \leq 0$$
$$h(x) = 0$$

***where x is an n-dimensional vector and g, h are vector functions. Data can be deterministic or stochastic and time-dependent.***

**Applications:1) Engineering design, including peta-scale methods, 2)Risk Management and Defend-Attack-Defend models, 3) Real-time logistics, 4) Optimal learning/machine learning, 5) Optimal photonic material design (meta-materials) , 6)Satellite and target tracking, 7)Embedded optimization for control, 8) Network data mining**

# Program Trends

**Analysis based optimization – Emphasis on theoretical results to exploit problem structure and establish convergence properties supporting the development of algorithms**

**Search based optimization – New methods to address very large continuous or discrete problems, but with an emphasis on the mathematical underpinnings, provable bounds, etc**

**Dynamic, stochastic and simulation optimization– New algorithms that address data dynamics and uncertainty and include optimization of simulation parameters**

**Combinatorial optimization – New algorithms that address fundamental problems in networks and graphs such as identifying substructures**

**Challenges: Curse of Dimensionality, Nonlinearity, Nondifferentiability, Uncertainty**

# Dynamic and Stochastic Methods

- **Stochastic optimization addresses the problem of making decisions over time as new information becomes available.**
- **Applications: operations research, economics, artificial intelligence and engineering.**
- **Fields: *Approximate dynamic programming, reinforcement learning, neuro-dynamic programming, optimal control and stochastic programming***

- **Dynamic Optimization**
  - **Powell, Sen, Magnanti and Levi**
    - Fleet scheduling, Refueling, ADP, Learning methods, Logistics
- **Stochastic Optimization –**
  - **Christos Cassandras**

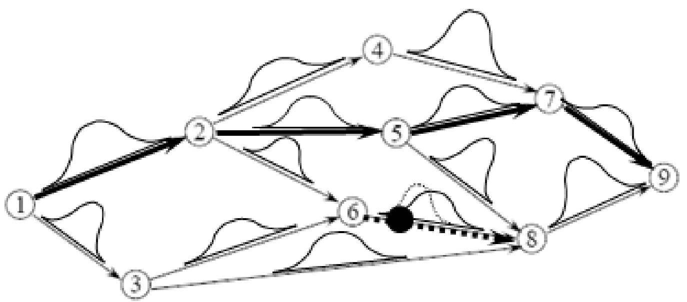
# Optimal learning and approximate dynamic programming

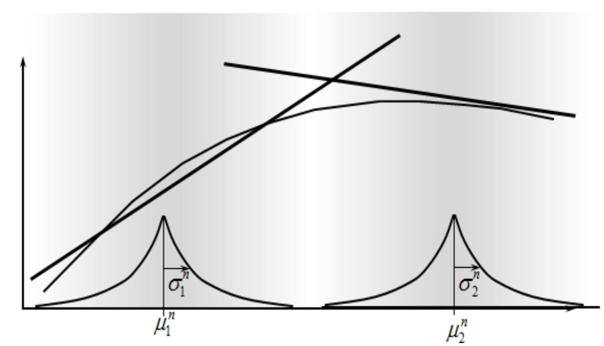
Warren B. Powell, Princeton University

**Goal:** To create a single, general purpose tool that will solve sequential decision problems for the broadest class of problems

**Challenges:** Complex resources, High-dimensional resource states, Complex state of the world variables, Belief States

- Optimal learning for general decision problems
  - Learning for linear programs – Optimal sequential learning for graphs and linear programs, dramatically expands the classical lookup table model.
  - Information blending – Generalizes learning problems to applications where information is blended; includes extension to robust objectives.
- Approximate dynamic programming
  - Fast, semiparametric approximation method avoids search for basis functions. Useful for stochastic search and ADP.
- Contributions at the learning/ADP interface
  - Learning with a physical state - bridges ADP and learning literature.
  - Knowledge gradient for semi-parametric beliefs

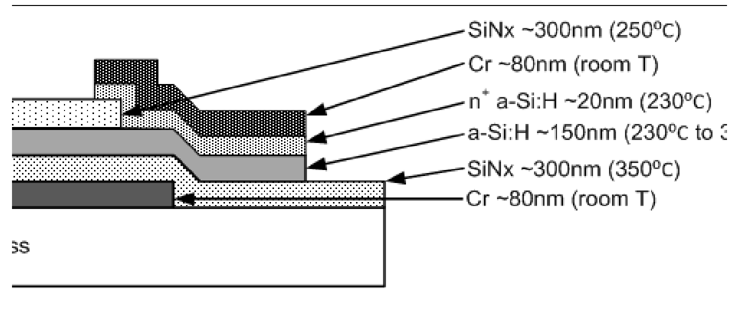


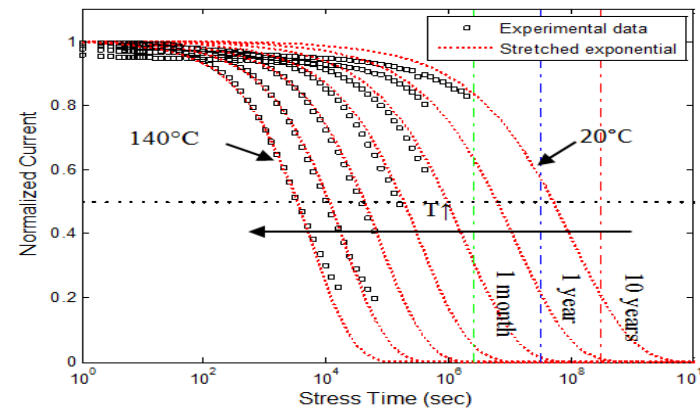



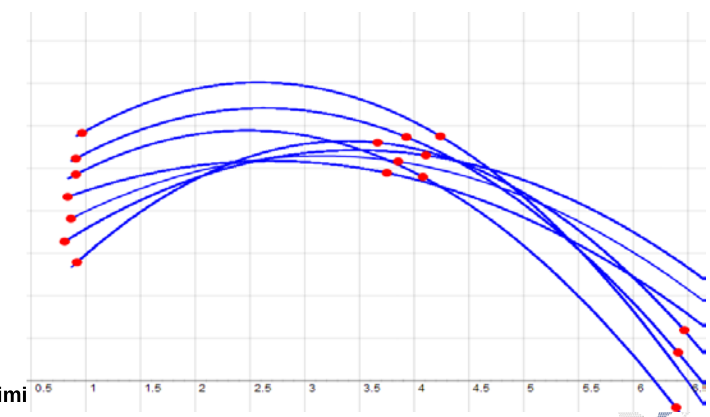
# Recent Transitions – 2012 BRI

**Warren B. Powell (Princeton University)**

- **Optimal learning for nano-bio technologies – Funded by AFOSR Natural Materials, Systems and Extremophiles Program (Hugh de Long)**
  - The goal of this research is to use optimal learning for the sequential design of experiments in the physical sciences, with a special focus on nano-bio.
  - Initial project addressing design of nanocrystalline silicon for high-performance/low-power transistor circuit technology on flexible substrates
  - Developing interactive tools for belief extraction which is used to compute knowledge gradient.



# OPTIMAL PERSISTENT SURVEILLANCE THROUGH COOPERATIVE TEAMS

C.G. Cassandras, Boston University

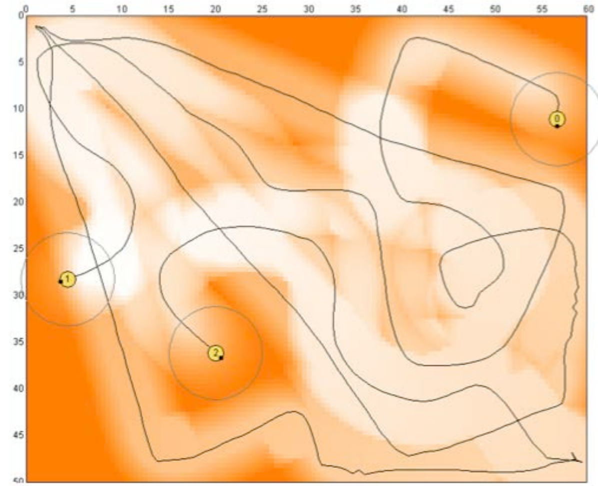
OPTIMAL PERSISTENT SURVEILLANCE PROBLEM:

Unlike the COVERAGE PROBLEM, the environment is too large to be fully covered by a stationary team of nodes (vehicles, sensors, agents). Instead:

– all areas of a mission space must be visited and sensed infinitely often

– some measure of overall uncertainty must be minimized

**What is the optimal cooperative way for nodes to move?**



Dark brown: HIGH uncertainty White: NO uncertainty

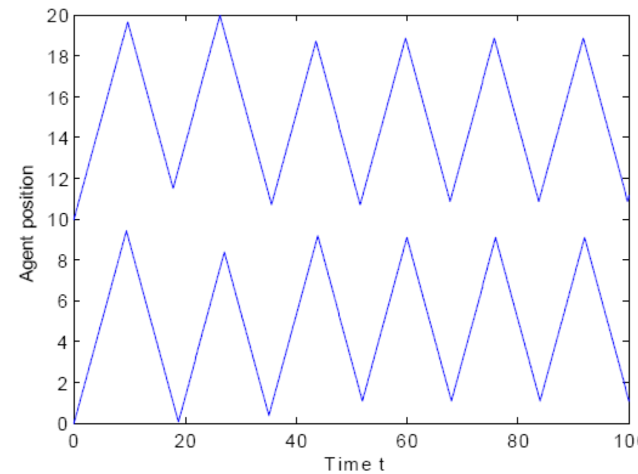
➡ KEY RESULT: In a one-dimensional mission space:

– optimal trajectories are oscillations between left and right direction-switching points

– nodes may dwell at these points for some time before reversing their motion

– these left and right points and associated dwell times can be efficiently determined

**Proofs based on combining Optimal Control and Infinitesimal Perturbation Analysis for Hybrid Systems**

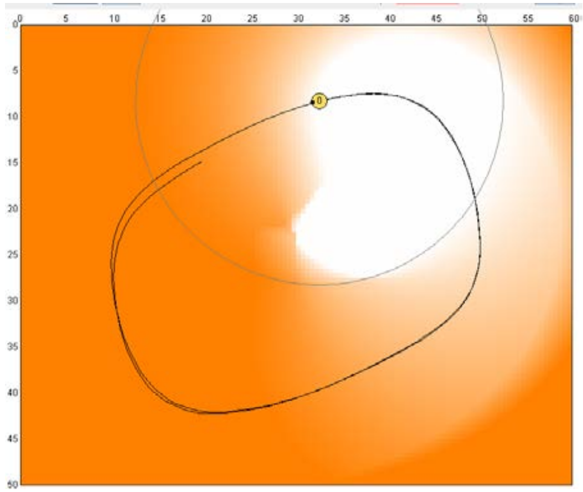



Optimal oscillatory trajectories of 2 nodes over a mission space [0, 20]

# OPTIMAL PERSISTENT SURVEILLANCE THROUGH COOPERATIVE TEAMS

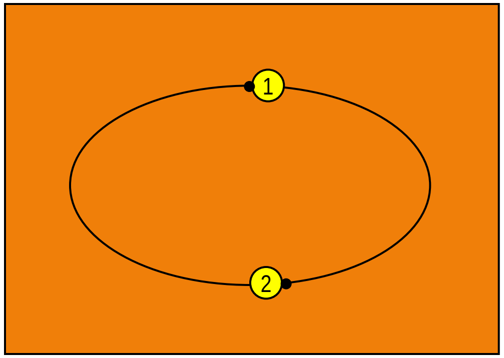
C.G. Cassandras, Boston University

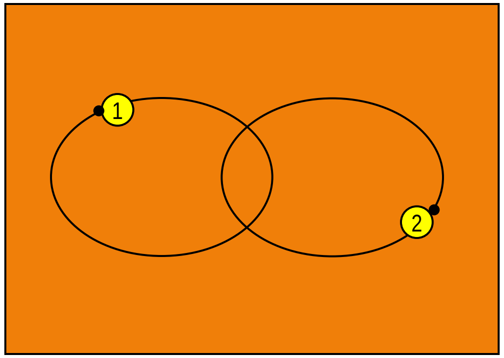
- KEY RESULT: In a two-dimensional mission space:
  - – single-node optimal trajectories are elliptical, not straight lines
  - – all ellipse parameters can be efficiently determined



**For two nodes, is a single ellipse with both nodes better or two ellipses with one node each?**

- PRELIMINARY RESULT: A single ellipse with two nodes outperforms two ellipses



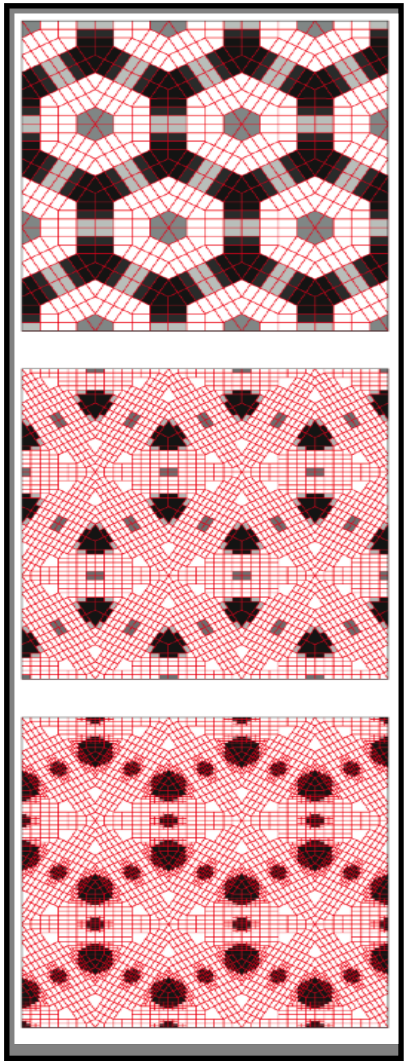



# Analysis-Based Optimization Theory and Methods

- **Mifflin and Sagastizabal**
  - *New theory leading to rapid convergence in non-differentiable optimization*
- **Freund and Peraire**
  - ***SDP for meta-material design***

# Optimization of photonic band gaps

## Freund & Peraire (MIT)



***Goal:*** Design of photonic crystals with multiple and combined band gaps

***Approach:*** *Solution Strategies*

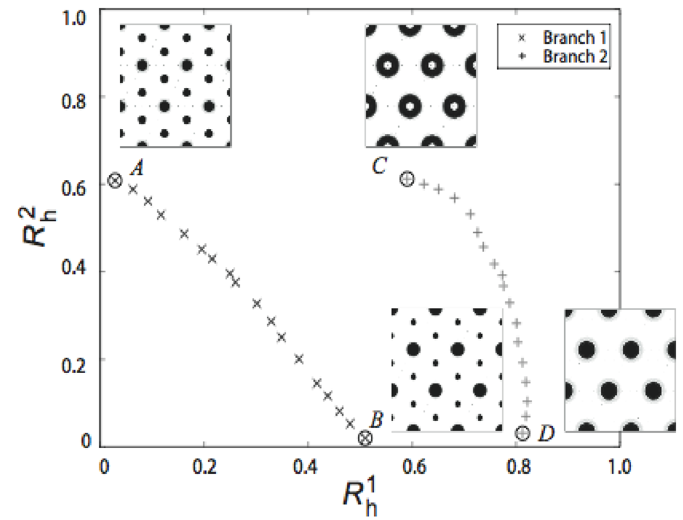
- Pixel-based topology optimization
- Large-scale non-convex eigen value optimization problem
- Relaxation and linearization

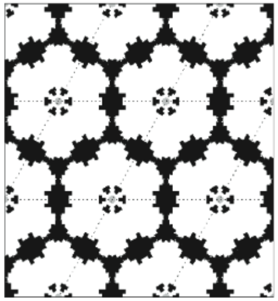
***Methods:*** *Numerical optimization tools*

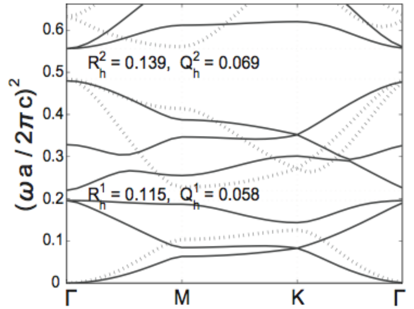
- Semi-definite programming
- Subspace approximation method
- Discrete system via Finite Element discretezation and Adaptive mesh refinement

**Results:** *Square and triangular lattices*

- **Very large absolute band gaps**
- **Multiple complete band gaps**
- **Much larger than existing results.**





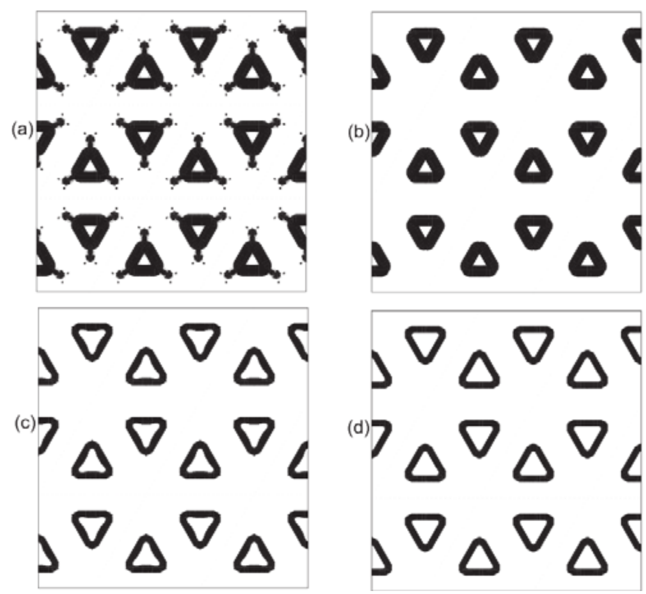
# Fabrication-Adaptive Optimization, Application to Meta-material Design

Freund and Peraire (MIT)

## Original Simple Optimization

## FA (fabrication-adaptive) Optimization

- FA optimization arises from the need to post-modify optimal solution x*, without severely compromising objective value.
- FA often leads to non-convex problems.
- Design strategy for photonic crystals: Nonlinear Program -> Semidefinite Program -> Linear Fractional Program.




(a) $x^*_O$ : original optimal design; $\text{Gap}(x^*_O) = 43.9\%$

(b) $y_O$: manual (5%) modification of $x^*_O$; $\text{Gap}(y_O) = 28.8\%$

(c) $x^*_{FA}$: FA optimal design with $\delta = 5\%$; $\text{Gap}(x^*_{FA}) = 34.2\%$

(d) $y_{FA}$: manual (0.8%) modification of $x^*_{FA}$; $\text{Gap}(y_{FA}) = 32.9\%$

** Submitted to Operations Research*

# Combinatorial Optimization

**Butenko (Texas A&M), Vavasis (U. Waterloo),**
**Krokhmal (U. Iowa), Prokopyev (U. Pittsburgh)**

*Underlying structures are often graphs; analysis is increasingly continuous non-linear optimization*

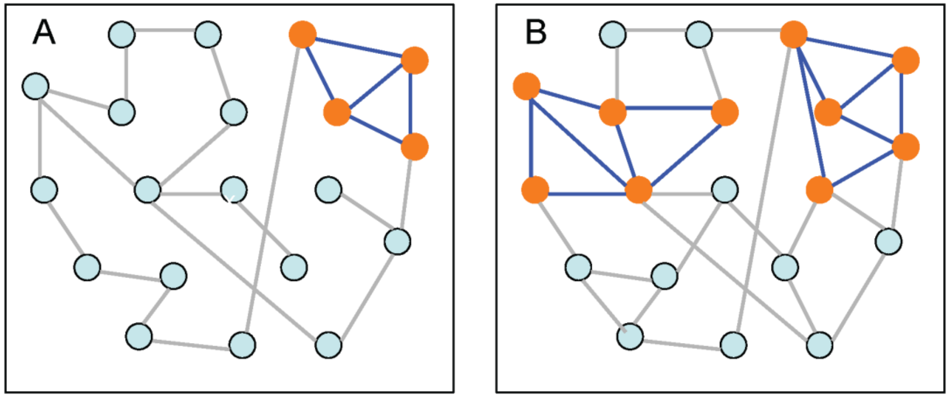
*Interested in detecting cohesive (tightly knit) groups of nodes representing clusters of the system's components: applications in acquaintance networks - Call networks, Protein interaction networks -Internet graphs*

*Phase Transitions in Random Graphs*

# First-Order Phase Transitions in Random Networks

P. Krokhmal (U. Iowa), V. Boginski, A. Veremyev (UFL), D. Jeffcoat (AFRL)

- Phase transitions are phenomena in large scale systems when system's properties change drastically once its size reaches certain threshold value
- Existing literature describes *continuous* (second-order) phase transitions in random graphs
- We report a *discontinuous* (first-order) phase transition **in the property of** *dense connectivity* of random graphs
- Densely connected components of graphs are modeled via *quasi-cliques*, or subgraphs of density at least $\gamma \leq 1$
- A popular $G(n,p)$ model of random graphs is used, where a link between two nodes exists with probability $0 \leq p \leq 1$




(A) A network with 20 nodes and 22 links. For $\gamma = 0.7$, the largest quasi-clique (at least 70%-dense subgraph) has size 4.

(B) Network from (A) where 10 more links have formed (e.g., due to increase of probability $p$). Highlighted are quasi-cliques with $\gamma = 0.7$. Their sizes are larger, but still significantly smaller than the size of the network

# First-Order Phase Transitions in Random Networks

P. Krokhmal (U. Iowa), V. Boginski, A. Veremyev (UFL), D. Jeffcoat (AFRL)

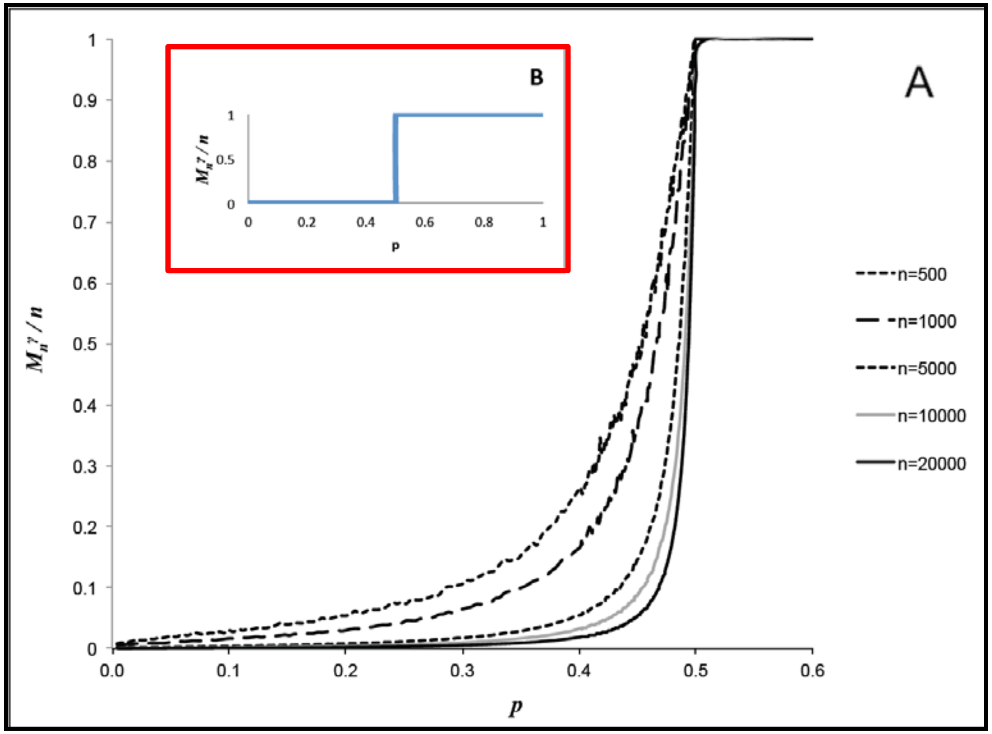
- As probability *p* of links in random *G*(*n*,*p*) graph increases, the size $M_n$ of the largest quasi-clique is log small in the graph size, as long as $p < \gamma$:

$$\frac{2\ln n}{\ln\left(\frac{1}{p}\right)} \le M_n \le \frac{2\ln n}{\ln\left(\left(\frac{\gamma}{p}\right)^{\gamma}\left(\frac{1-\gamma}{1-p}\right)^{1-\gamma}\right)} \quad \text{a.s.}$$

- But, $M_n$ undergoes *discontinuous* jump (*first-order phase transition*) at $p = \gamma$:

$$\boxed{\begin{aligned} \lim_{p \nearrow \gamma}\lim_{n\to\infty}\frac{M_n}{n} &= 0, \quad \text{w.h.p.} \\ \lim_{p \searrow \gamma}\lim_{n\to\infty}\frac{M_n}{n} &= 1, \quad \text{w.h.p.} \end{aligned}}$$

- This is one of first 1st-order phase transitions in uniform random graphs reported in the literature




(A) Relative sizes of quasi-cliques in small-to-average-sized random graphs (note a continuous transition becoming more abrupt as graph size increases)

(B) Relative sizes of quasi-cliques in extremely large random graphs (note the discontinuous jump)

AV, VB, PK, DJ "Dense percolations in large-scale mean-field networks is provably explosive", *PLOS ONE* (2012)

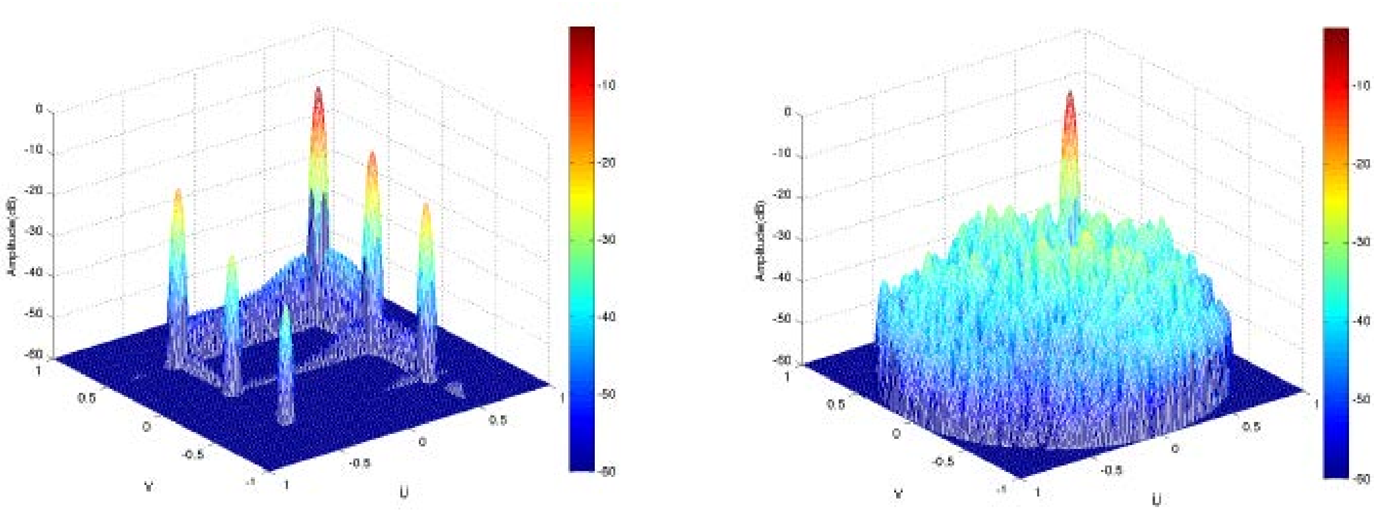
# Polyomino Tiling

O. Prokopyev (U. Pittsburgh)

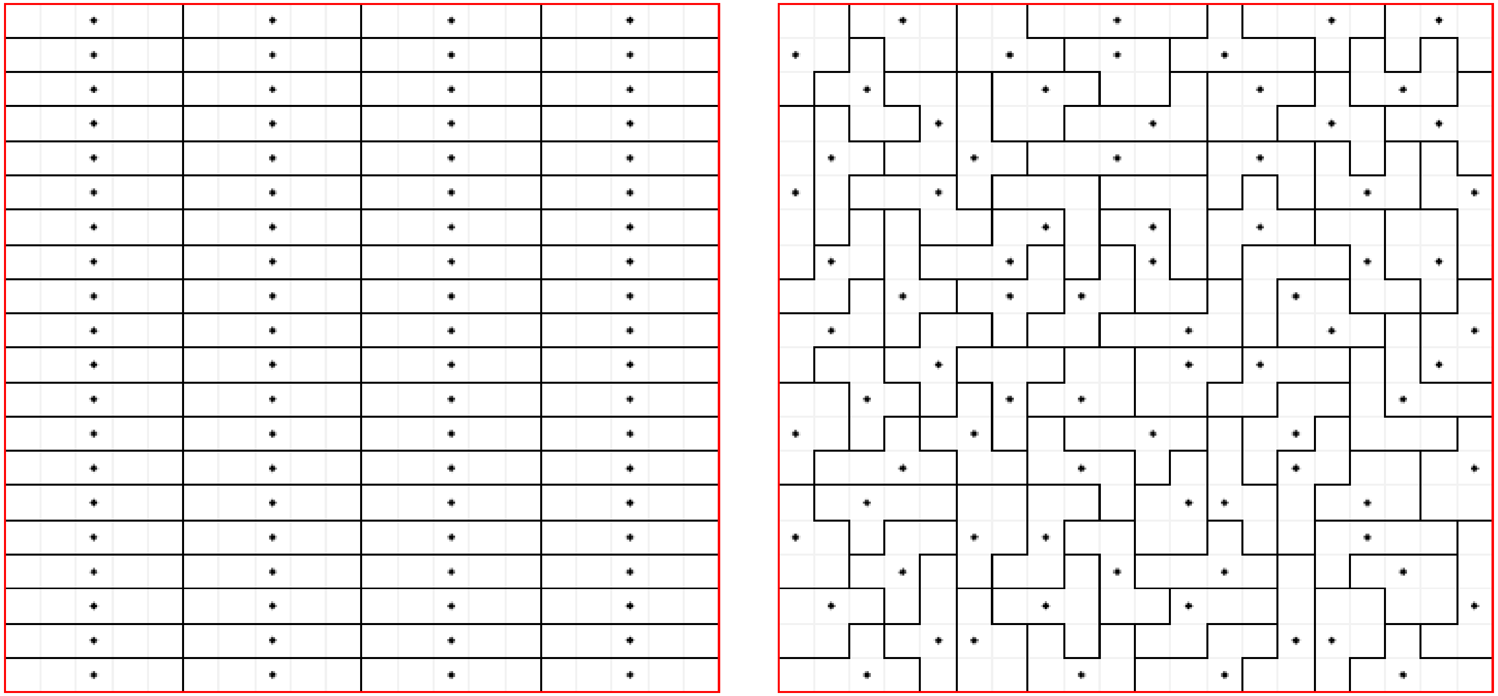
- **Application:**
  - **Time delay control of phased array systems**
- **Jointly supported with Dr. Nachman of AFOSR/RSE, motivated by work at the Sensors Directorate:**
  - ***Irregular Polyomino-Shaped Subarrays for Space-Based Active Arrays*** (R.J. Mailloux, S.G. Santarelli, T.M. Roberts, D. Luu)

# Optimization for Irregular Polyomino Tilings

- Motivation: Mailloux et al. [2006, 2009] showed that a phased array of "irregularly" packed polyomino shaped subarrays has peak quantization sidelobes suppressed compared to an array with rectangular periodic subarrays
- Sample simulation: rectangular (regular) subarray vs. irregular subarray - peak sidelobes are significantly reduced



- Goal: arrange polyomino-shaped subarrays to maximize "irregularity"
- Approach: nonlinear mixed integer programming (NMIP), where "irregularity" is modeled based on information theoretic entropy



- **Example:** rectangular (regular) tiling vs. irregular 20x20 tiling using pentomino family (each shape contains exactly 5 unit squares)
- "Irregularity" is measured using information theoretic entropy: minimum vs. maximum entropy in the example above
- **Results:** we developed exact and approximate solution methods
  - ✓ exact methods – solve instances of size up to 80x100 in a reasonable time (it depends on polyomino family)
  - ✓ heuristic and approximate methods - thousands (limitations mainly on storage memory)

# Summary

- **Goals :**
  - Cutting edge optimization research
  - Of future use to the Air Force
- **Strategies :**
  - Stress rigorous models and algorithms , especially for environments with rapidly evolving and uncertain data
  - Engage more recognized optimization researchers, including more young PIs
  - Foster collaborations –
    - Optimizers, engineers and scientists
    - Between PIs and AF labs
    - With other AFOSR programs
    - With NSF, ONR, ARO